8.1

The Chain Rule

Learning objectives

- To discuss the chain rule, the rule for differentiation of composite functions

  And

- To practice related problems.

The Chain Rule

We have learnt how to differentiate simple functions like $\sin x$ and $(x^2 - 4)$. Now we extend the differentiation to composites like $sin\ (x^2 - 4)$ by using the Chain Rule.

The Chain Rule says that *the derivative of the composite of two differentiable functions is the product of their derivatives evaluated at appropriate points*. The Chain Rule is probably the most widely used differentiation rule in calculus. This module describes the rule and explains how to use it. First, we consider a few examples.

Example 1: The function $y = 6x - 10 = 2(3x - 5)$ is the composite of the functions $y = 2u$ and $u = 3x - 5$. How are the derivatives of these three functions related?

Solution:

We have, $\frac{dy}{dx} = 6$

$$\frac{dy}{du} = 2, \quad \frac{du}{dx} = 3$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

If we think of the derivative as a rate of change, this relationship seems to be reasonable. For, $y = f(u)$ and $u = g(x)$, if y changes twice as fast as u and u changes three times as fast as x, then we expect y to change six times as fast as x.

Theorem

If $f(u)$ is differentiable at the point $u = g(x)$, and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f\left(g(x)\right)$ is differentiable at x, and

$$(f \circ g)'(x) = f'\left(g(x)\right) \cdot g'(x) \qquad \text{... (1)}$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \text{... (2)}$$

where $\frac{dy}{du}$ is evaluated at $u = g(x)$.

Example 3

Find the derivative of $y = \sqrt{x^2 + 1}$.

Solution: Here $y = f\left(u(x)\right)$**, where** $f(u) = \sqrt{u}$ **and** $u = x^2 + 1$**.**

Since the derivatives f and g are

$$f'(u) = \frac{1}{2\sqrt{u}},\ u'(x) = 2x,$$

the Chain Rule gives

$$\frac{dy}{dx} = \frac{d}{dx} f\left(u(x)\right) = f'\left(u(x)\right) \cdot u'(x)$$

$$= \frac{1}{2\sqrt{u(x)}} \cdot u'(x) = \frac{1}{2\sqrt{x^2+1}} \cdot (2x) = \frac{x}{\sqrt{x^2+1}}$$

P1:

Find the derivative of the function given by

$$\boldsymbol{f(x) = sin(3x - 5)}$$

Solution:

Given, $f(x) = \sin(3x - 5)$,

now $f(x) = \sin u$,where $u = 3x - 5$

By chain rule

$$\frac{df}{dx} = \frac{df}{du} . \frac{du}{dx}$$

$$= \frac{d}{du}(sinu). \frac{d}{dx}(3x - 5) = \cos\ u. 3$$

Thus $\frac{df}{dx} = 3\cos\ (3x - 5)$

P2:

Find the derivative of the function given by

$$f(x) = \sin\left(cos(x^2)\right)$$

Solution:

Given, $f(x) = \sin(cosx^2)$

The given function is a composite of three functions.

If $f(x) = \sin v, v(w) = cos\, w\ and\ w(x) = x^2$

$\frac{df}{dv} = cosv$, $\frac{dv}{dw} = -sinw$ and $\frac{dw}{dx} = 2x$.

Hence, by chain rule

$$\frac{df}{dx} = \frac{df}{dv}.\frac{dv}{dw}.\frac{dw}{dx} = cosv.(-sinw).2x$$

$$= -2xcos(x^2).sin(x^2)$$

Thus $\frac{df}{dx} = -2xcos(cosx^2).sin(x^2)$

P3:

The derivative of $\boldsymbol{y = sin(x - cosx)}$

Solution:

$y = sin(x - cosx)$

$y = sinu$, where $u(x) = x - cosx$

By chain rule

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx} = \frac{d}{du}(sinu) . \frac{d}{dx}(x - cosx)$$

$$= cosu . (1 + sinx) = (1 + sinx)cos(x - cosx).$$

P4:

Differentiate $\boldsymbol{y = \sin(cos\,(2t-5))}$

Solution:

$$y = \sin(cos\,(2t-5))$$

$y = sinu, u = cosv$, where $v = (2t-5)$

By chain rule

$$\frac{dy}{dt} = \frac{dy}{du} . \frac{du}{dv} . \frac{dv}{dt}$$

$$= \frac{d(sinu)}{du} . \frac{d(cosv)}{dv} . \frac{d(2t-5)}{dt}$$

$$= cosu.(-sinv).2 = -2cos(cos\,(2t-5))\sin(2t-5)$$

IP1:

Find the derivative of the function given by

$$\boldsymbol{f(x) = \sec(tanx)}$$

Solution:

Given, $\sec(tanx)$

$f(x) = \sec(tanx)$

$f(x) = secu$, where $u(x) = tanx$

$$\frac{df}{du} = secu\ tanu\ and\ \frac{du}{dx} = sec^2x$$

By chain rule,

$$\frac{df}{dx} = \frac{df}{du}.\frac{du}{dx} = secu\ tanu.sec^2x$$

$$= \sec(tanx)\ \ tan(tanx).sec^2x$$

Thus $\frac{df}{dx} = \sec(tanx)\ \ tan(tanx).sec^2x$

IP2:

If $\boldsymbol{f(x) = sin(cos(tanx))}$, then find $\boldsymbol{f'(x)}$

Solution:

$$f(x) = sin(cos(tanx))$$

$f(x) = \sin\, u, u(v) = cosv$ and $v(x) = tanx$

$\frac{df}{du} = cosu$, $\frac{du}{dv} = -sinv$ and $\frac{dv}{dx} = sec^2x$

By chain rule,

$$\frac{df}{dx} = \frac{df}{du}.\frac{du}{dv}.\frac{dv}{dx} = -cosu.(sinv).sec^2x$$

$$= -cos(cos(tanx)).(sin(tanx)).sec^2x$$

$$\frac{df}{dx} = -cos(cos(tanx)).(sin(tanx)).sec^2x$$

IP3:

The derivative of $\boldsymbol{y = cot\left(\frac{sint}{t}\right)}$

Solution:

$$y = cot\left(\frac{sint}{t}\right)$$

$y = cotu$, where $u(t) = \frac{sint}{t}$

By chain rule

$$\frac{dy}{dt} = \frac{dy}{du}.\frac{du}{dt} = \frac{d}{du}(cotu).\frac{d}{dt}\left(\frac{sint}{t}\right)$$

$$= -csc^2u.\left(\frac{t.cost - sint}{t^2}\right) = -\left(\frac{t.cost - sint}{t^2}\right)csc^2\left(\frac{sint}{t}\right).$$

IP4:

Differentiate $\boldsymbol{y = \cos\left(5sin\left(\frac{t}{5}\right)\right)}$**:**

Solution:

$$y = \cos\left(5\sin\left(\frac{t}{5}\right)\right)$$

$y = cosu, u = 5sinv$, where $v = \frac{t}{5}$

By chain rule

$$\frac{dy}{dt} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dt}$$

$$= \frac{d}{du}(cosu).\frac{d}{dv}(5sinv).\frac{d}{dt}\left(\frac{t}{5}\right)$$

$$= -sinu.(5cosv).\frac{1}{5} = -sin\left(5\sin\left(\frac{t}{5}\right)\right)\cos\left(\frac{t}{5}\right)$$

8.1. The Chain Rule

Exercise:

I. Given $y = f(u)$ *and* $u = g(x)$, find $\frac{dy}{dx} = f'(g(x))g'(x)$

1. $y = sinu, u = 3x + 1$
2. $y = cosu, u = -\frac{x}{3}$
3. $y = cosu, u = sinx$
4. $y = sinu, u = x - cosx$
5. $y = tanu, u = 10x - 5$

II. Find the derivative of the functions below:

1. $s = \frac{4}{3\pi} sin3t + \frac{4}{5\pi} cos5t$
2. $s = sin\left(\frac{3\pi t}{2}\right) + cos\left(\frac{3\pi t}{2}\right)$
3. $y = \sin(\cos(2t - 5))$
4. $y = cos\left(5sin\left(\frac{t}{3}\right)\right)$
5. $y = \sec(tanx)$
6. $y = \cot\left(\pi - \frac{1}{x}\right)$

8.2

Differentiation formulas that includes chain rule

Learning objectives:

- To study the power chain rule

  And

- To solve related problems

Many of the differentiation formulas we encounter already include the Chain Rule. We illustrate this as shown below.

If f is a differentiable function of u, and u is a differentiable function of x, then substituting $y = f(u)$ in the Chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

leads to the formula

$$\frac{d}{dx} f(u) = f'(u) \frac{du}{dx}$$

For example, if u is a differentiable function of x, n is an integer, and $y = u^n$, then the Chain Rule gives

$$\frac{dy}{dx} = \frac{d}{du}\left(u^n\right) \cdot \frac{du}{dx}$$

$$= nu^{n-1} \frac{du}{dx}$$

Power Chain Rule

If $u(x)$ is a differentiable function and n is an integer, then u^n is differentiable and

$$\frac{d}{dx} u^n = nu^{n-1} \frac{du}{dx}$$

Example 1:

a) $\frac{d}{dx}\sin^5 x = 5\sin^4 x\frac{d}{dx}\sin x = 5\sin^4 x\cos x$

b) $\frac{d}{dx}(2x+1)^{-3} = -3(2x+1)^{-4}\frac{d}{dx}(2x+1) = -6(2x+1)^{-4}$

c) $\frac{d}{dx}(5x^3 - x^4)^7 = 7(5x^3 - x^4)^6\frac{d}{dx}(5x^3 - x^4) = 7(5x^3 - x^4)^6(15x^2 - 4x^3)$

d) $\frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x-2)^{-1} = (-1)(3x-2)^{-2}(3) = -\frac{3}{(3x-2)^2}$

Example 2:

It is important to remember that the formulas for the derivatives of $\sin x$ and $\cos x$ were obtained under the assumption that x *is measured in radians, not in degrees*.

Since $180° = \pi$ radians, $x° = \frac{\pi x}{180}$ radians.

By the Chain Rule,

$$\begin{aligned}\frac{d}{dx}\sin(x^\circ) &= \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right)\\ &= \cos\left(\frac{\pi x}{180}\right).\frac{d}{dx}\left(\frac{\pi x}{180}\right)\\ &= \frac{\pi}{180}\cos(x^\circ)\end{aligned}$$

P1:

Find the derivative of the function $\boldsymbol{y = \left(\frac{x}{5} + \frac{1}{5x}\right)^5}$

Solution:

$$y = \left(\frac{x}{5} + \frac{1}{5x}\right)^5 \Rightarrow y = (u)^5, \text{ where } u(x) = \left(\frac{x}{5} + \frac{1}{5x}\right)$$

By chain rule

$$\frac{dy}{dx} = y'(u).\frac{du}{dx}, \text{ where } \frac{dy}{du} = 5.u^4 \text{ and } \frac{du}{dx} = \left(\frac{1}{5} - \frac{1}{5x^2}\right)$$

$$\therefore \frac{dy}{dx} = 5\left(\frac{x}{5} + \frac{1}{5x}\right)^4.\left(\frac{1}{5} - \frac{1}{5x^2}\right)$$

P2:

Find the derivative of the function $\boldsymbol{s = sin^2\left(\frac{3\pi t}{2}\right) - cos^2\left(\frac{3\pi t}{2}\right)}$

Solution:

$$s = sin^2\left(\frac{3\pi t}{2}\right) - cos^2\left(\frac{3\pi t}{2}\right)$$

$$\frac{ds}{dt} = \frac{d}{dt}\left(sin^2\left(\frac{3\pi t}{2}\right)\right) - \frac{d}{dt}\left(cos^2\left(\frac{3\pi t}{2}\right)\right)$$

$$= 2sin\left(\frac{3\pi t}{2}\right)\frac{d}{dt}\left(sin\left(\frac{3\pi t}{2}\right)\right) - 2cos\left(\frac{3\pi t}{2}\right)\frac{d}{dt}\left(cos\left(\frac{3\pi t}{2}\right)\right)$$

$$= 2sin\left(\frac{3\pi t}{2}\right) cos\left(\frac{3\pi t}{2}\right)\frac{d}{dt}\left(\frac{3\pi t}{2}\right) + 2cos\left(\frac{3\pi t}{2}\right) sin\left(\frac{3\pi t}{2}\right)\frac{d}{dt}\left(\frac{3\pi t}{2}\right)$$

$$= 2sin\left(\frac{3\pi t}{2}\right) cos\left(\frac{3\pi t}{2}\right)\left(\frac{3\pi}{2}\right) + 2cos\left(\frac{3\pi t}{2}\right) sin\left(\frac{3\pi t}{2}\right)\left(\frac{3\pi}{2}\right)$$

$$= \left(\frac{3\pi}{2}\right)\left[2sin\left(\frac{3\pi t}{2}\right) cos\left(\frac{3\pi t}{2}\right) + 2cos\left(\frac{3\pi t}{2}\right) sin\left(\frac{3\pi t}{2}\right)\right]$$

$$= 3\pi\left[2sin\left(\frac{3\pi t}{2}\right) cos\left(\frac{3\pi t}{2}\right)\right] = 3\pi\left[sin\left(2.\frac{3\pi t}{2}\right)\right] = 3\pi \sin\,(3\pi t)$$

P3:

Find the derivative of $\boldsymbol{y = \left(1 + tan^4\left(\frac{t}{12}\right)\right)^3}$

Solution:

Given $y = \left(1 + tan^4\left(\frac{t}{12}\right)\right)^3$

$$\frac{dy}{dt} = \frac{d}{dt}\left(1 + tan^4\left(\frac{t}{12}\right)\right)^3$$

$$= 3\left(1 + tan^4\left(\frac{t}{12}\right)\right)^2 \frac{d}{dt}\left(1 + tan^4\left(\frac{t}{12}\right)\right)$$

$$= 3\left(1 + tan^4\left(\frac{t}{12}\right)\right)^2 \left(0 + 4tan^3\left(\frac{t}{12}\right)\frac{d}{dt}\left(tan\left(\frac{t}{12}\right)\right)\right)$$

$$= 3\left(1 + tan^4\left(\frac{t}{12}\right)\right)^2 \left(4tan^3\left(\frac{t}{12}\right)sec^2\left(\frac{t}{12}\right)\frac{d}{dt}\left(\frac{t}{12}\right)\right)$$

$$= 3\left(1 + tan^4\left(\frac{t}{12}\right)\right)^2 \left(4tan^3\left(\frac{t}{12}\right)sec^2\left(\frac{t}{12}\right)\frac{1}{12}\right)$$

$$= \left(1 + tan^4\left(\frac{t}{12}\right)\right)^2 \left(tan^3\left(\frac{t}{12}\right)sec^2\left(\frac{t}{12}\right)\right)$$

P4:

If $y = (x^3 - 1)^{100}$ then find $\frac{dy}{dx}$.

Solution:

Given $y = (x^3 - 1)^{100}$

$y = u^{100}(x)$, where $u(x) = x^3 - 1$

By chain rule,

$\frac{dy}{dx} = y'(u).u'(x)$, where $\frac{dy}{dx} = 100.u^{99}$, $u'(x) = 3x^2$

$$\frac{dy}{dx} = 100(x^3 - 1)^{99} \times 3x^2 = 300x^2(x^3 - 1)^{99}$$

IP1:

Find the derivative of the function $\boldsymbol{y = sin^3x}$

Solution:

$y = sin^3x \Longrightarrow y = u^3$, Where $u(x) = sinx$

By chain rule,

$\frac{dy}{dx} = y'(u).\frac{du}{dx}$, where $\frac{dy}{du} = 3(u)^2\ and\ \frac{du}{dx} = cosx$

Now

$$\frac{dy}{dx} = 3(sinx)^2cosx = 3sin^2xcosx = \frac{3}{2}\ sin2x.sinx$$

$$\therefore \frac{dy}{dx} = \frac{3}{2}\ sin2x.sinx$$

IP2:

Find the derivative of the function $\boldsymbol{y = \frac{1}{6}\left(1 + cos^2(7t)\right)^3}$

Solution:

$$y = \frac{1}{6}\left(1 + cos^2(7t)\right)^3$$

$$\frac{dy}{dt} = \frac{1}{6}\frac{d}{dt}\left(1 + cos^2(7t)\right)^3$$

$$= \frac{1}{6}.3.\left(1 + cos^2(7t)\right)^2 \frac{d}{dt}\left(1 + cos^2(7t)\right)$$

$$= \frac{1}{2}.\left(1 + cos^2(7t)\right)^2 \left(0 + 2cos(7t)\frac{d}{dt}\left(cos(7t)\right)\right)$$

$$= \frac{1}{2}.\left(1 + cos^2(7t)\right)^2 \left(2cos(7t).\left(-sin(7t)\right)\frac{d}{dt}(7t)\right)$$

$$= -\frac{7}{2}.\left(1 + cos^2(7t)\right)^2 sin14t$$

IP3:

Find the derivative of $\boldsymbol{y = 4sin^2\left(\sqrt{1+\sqrt{t}}\right)}$

Solution:

$$y = 4sin^2\left(\sqrt{1+\sqrt{t}}\right)$$

$$\frac{dy}{dt} = 4\frac{d}{dt}\left(sin^2\left(\sqrt{1+\sqrt{t}}\right)\right)$$

$$= 4.2.\,sin\left(\sqrt{1+\sqrt{t}}\right)\frac{d}{dt}\left(sin\left(\sqrt{1+\sqrt{t}}\right)\right)$$

$$= 4.2.\,sin\left(\sqrt{1+\sqrt{t}}\right)cos\left(\sqrt{1+\sqrt{t}}\right)\frac{d}{dt}\left(\sqrt{1+\sqrt{t}}\right)$$

$$= 4.\,sin\left(2\sqrt{1+\sqrt{t}}\right)\frac{1}{2\left(\sqrt{1+\sqrt{t}}\right)}\frac{d}{dt}(1+\sqrt{t})$$

$$= 4.\,sin\left(2\sqrt{1+\sqrt{t}}\right)\frac{1}{2\left(\sqrt{1+\sqrt{t}}\right)}\cdot\frac{1}{2\sqrt{t}}$$

$$= \frac{sin\left(2\sqrt{1+\sqrt{t}}\right)}{\sqrt{t}\left(\sqrt{1+\sqrt{t}}\right)}$$

IP4:

Differentiate $\boldsymbol{f(y) = \left(\frac{y^2}{y+1}\right)^5}$

Solution:

Given, $f(y) = \left(\frac{y^2}{y+1}\right)^5$

Let, $f(u) = u^5$, where $u(y) = \frac{y^2}{y+1}$

By chain rule,

$f'(y) = f'(u).u'$

where $f'(u) = 5.u^4$,

$$u'(y) = \frac{(y+1)(2y) - y^2(1)}{(y+1)^2} = \frac{2y^2 + 2y - y^2}{(y+1)^2} = \frac{2y - y^2}{(y+1)^2}$$

$f'(y) = f'(u).u'(y)$

$$= 5.\left(\frac{y^2}{y+1}\right)^4 . \frac{(2y - y^2)}{(y+1)^2}$$

8.2.Differentiation formulas that includes chain rule

Exercise

I. **Find the derivative of the following functions:**

a. $y = (2x + 1)^5$

b. $y = (4 - 3x)^9$

c. $y = \left(1 - \frac{x}{7}\right)^{-7}$

d. $y = \left(\frac{x}{2} - 1\right)^{-10}$

e. $y = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^4$

f. $y = \left(\frac{x}{5} + \frac{1}{5x}\right)^5$

g. $y = \frac{1}{21}(3x - 2)^7 + \left(4 - \frac{1}{2x^2}\right)^{-1}$

h. $y = (5 - 2x)^{-3} + \frac{1}{8}\left(\frac{2}{x} + 1\right)^4$

i. $y = (4x + 3)^4(x + 1)^{-3}$

j. $y = (2x - 5)^{-1}(x^2 - 5x)^6$

II. Find the derivative of the following functions:

a. $f(\theta) = \left(\frac{sin\theta}{1+cos\theta}\right)^2$

b. $g(t) = \left(\frac{1+cost}{sint}\right)^{-1}$

c. $y = sin^2(\pi t - 2)$

d. $y = sec^2 \pi t$

e. $y = (1 + cos2t)^{-4}$

f. $y = \left(1 + cot\left(\frac{t}{2}\right)\right)^{-2}$

g. $y = sin^3 x$

h. $y = 5\, cos^{-4} x$

i. $y = \left(1 + tan^4\left(\frac{t}{12}\right)\right)^3$

j. $y = \frac{1}{6}\left(1 + cos^2(7t)\right)^3$

k. $r = (csc\,\theta + cot\theta)^{-1}$

l. $r = -(sec\theta + tan\theta)^{-1}$

m. $y = x^2 sin^4 x + x cos^{-2} x$

n. $y = \frac{1}{x} sin^{-5} x - \frac{x}{3} cos^3 x$

8.4

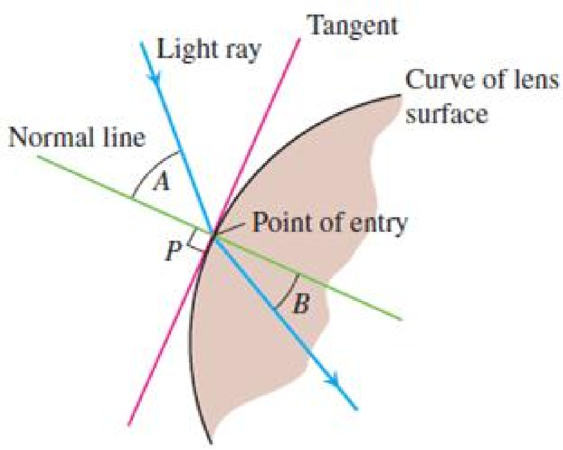
Tangents and Normal Lines

Learning objectives:

- To find the tangent and the normal lines of a given curve at a given point.
- To find higher order derivatives using implicit differentiation
  And
- To practice related problems.

Tangents and Normal Lines

In the law that describes how light changes direction as it enters a lens, the important angles are the angles the light makes with the line perpendicular to the surface of the lens at the point of entry (angles A and B in figure below). This line is called the *normal* to the surface at the point of entry.



The above figure is a profile view of a lens and the normal is the line perpendicular to the tangent to the profile curve at the point of entry.

Definition

A line is normal to a curve at a point if it is perpendicular to the curve's tangent there. The line is called the ***normal*** to the curve at that point.

The profiles of lenses are often described by quadratic curves. When they are, we can use implicit differentiation to find the tangents and normals.

Example 1: Find the tangent and normal to the curve $\boldsymbol{x^2 - xy + y^2 = 7}$ at the point $(-1, 2)$.

Solution: We first use implicit differentiation to find $\frac{dy}{dx}$:

$$x^2 - xy + y^2 = 7$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(7)$$

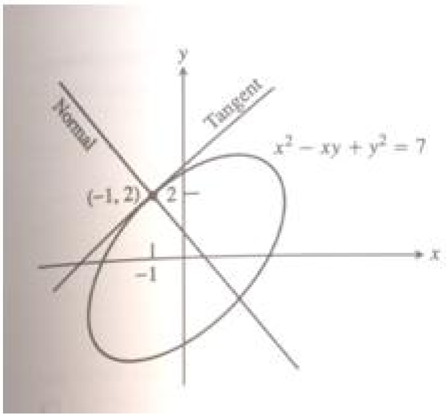
$$2x - \left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) + 2y\frac{dy}{dx} = 0$$

$$(2y - x)\frac{dy}{dx} = y - 2x$$

$$\frac{dy}{dx} = \frac{y-2x}{2y-x}$$

We then evaluate the derivative at $(x, y) = (-1, 2)$ to obtain

$$\left.\frac{dy}{dx}\right|_{(-1,2)} = \left.\frac{y-2x}{2y-x}\right|_{(-1,2)} = \frac{2-2(-1)}{2(2)-(-1)} = \frac{4}{5}$$



The tangent to the curve at $(-1, 2)$ is the line

$$y = 2 + \frac{4}{5}(x - (-1))$$

$$y = \frac{4}{5}x + \frac{14}{5}$$

The normal to the curve at $(-1, 2)$ is

$$y = 2 - \frac{5}{4}(x - (-1))$$

$$y = -\frac{5}{4}x + \frac{3}{4}$$

Using Implicit Differentiation to find derivatives of Higher Order

Implicit differentiation can also be used to calculate higher order derivatives.

Example 2: Find $\frac{d^2y}{dx^2}$ if $\mathbf{2x^3 - 3y^2 = 7}$.

Solution: We differentiate both sides of the equation with respect to $\boldsymbol{x}$ to find $\boldsymbol{y}' = \frac{dy}{dx}$:

$$2x^3 - 3y^2 = 7$$

$$\Rightarrow \frac{d}{dx}(2x^3) - \frac{d}{dx}(3y^2) = \frac{d}{dx}(7)$$

$$6x^2 - 6\,yy' = 0 \Rightarrow x^2 - yy' = 0 \Rightarrow y' = \frac{x^2}{y} \quad (\text{If } y \neq 0)$$

We differentiate $x^2 - yy' = 0$ again to find y'':

$$\frac{d}{dx}(x^2 - yy') = \frac{d}{dx}(0)$$

$$\Rightarrow 2x - yy'' - y'y' = 0 \Rightarrow yy'' = 2x - (y')^2$$

$$\Rightarrow y'' = \frac{2x}{y} - \frac{(y')^2}{y} \qquad (y \neq 0)$$

Finally, we substitute $y' = \frac{x^2}{y}$ to express y'' in terms of x and y:

$$y'' = \frac{2x}{y} - \frac{\left(\frac{x^2}{y}\right)^2}{y} = \frac{2x}{y} - \frac{x^4}{y^3}, \qquad (y \neq 0)$$

P1:

Find the tangent and normal to the curve $x^2 + y^2 = y^4 - 2x$ at $(-2, 1)$.

Solution:

The given curve is $x^2 + y^2 = y^4 - 2x$ and $(-2,1)$ lies on it.

Differentiate with respect to x

$$2x + 2y\frac{dy}{dx} = 4y^3\frac{dy}{dx} - 2$$

$$\Longrightarrow (2y - 4y^3)\frac{dy}{dx} = -2 - 2x$$

$$\Longrightarrow \frac{dy}{dx} = \frac{1+x}{2y^3-y}$$

$$\therefore \text{slope} = \frac{dy}{dx}_{(-2,1)} = \frac{1+(-2)}{2(1)-1} = \frac{-1}{1} = -1$$

Tangent to the curve at $(-2, 1)$

$$y - 1 = (-1)(x + 2) \Longrightarrow x + y + 1 = 0$$

Slope of the normal to the curve at $(-2, 1) = -\frac{1}{\left(\frac{dy}{dx}\right)_{(-2,1)}} = 1$

Normal to the curve at $(-2, 1)$

$$y - (1) = 1(x + 2) \Longrightarrow x - y + 3 = 0$$

P2:

Find the tangent and normal to the curve $x\ sin2y = y\ cos2x$ at $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

Solution:

The given curve is $xsin2y = y\ cos2x$ and $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ lies on it.

Differentiate both sides with respect to x

$$sin2y + x.cos2y.2.\frac{dy}{dx} = y.(-sin2x).2 + cos2x.\frac{dy}{dx}$$

$$\Rightarrow (2x\ cos2y - cos2x)\frac{dy}{dx} = -2y\ sin2x - sin2y$$

$$\Rightarrow \frac{dy}{dx} = \frac{-[sin2y + 2y\ sin2x]}{2x\ cos2y - cos2x}$$

$$\therefore \text{slope} = \frac{dy}{dx}_{\left(\frac{\pi}{4}, \frac{\pi}{2}\right)} = \frac{-[0+\pi.(1)]}{2\times\frac{\pi}{4}(-1)-0} = \frac{-\pi}{-\frac{\pi}{2}} = 2$$

Tangent to the curve at $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$$y - \frac{\pi}{2} = 2\left(x - \frac{\pi}{4}\right) \Rightarrow 4x - 2y = 0 \Rightarrow 2x - y = 0$$

Slope of the normal to the curve at $\left(\frac{\pi}{4}, \frac{\pi}{2}\right) = -\frac{1}{\left(\frac{dy}{dx}\right)_{\left(\frac{\pi}{4},\frac{\pi}{2}\right)}} = -\frac{1}{2}$

Normal to the curve at $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$$y - \frac{\pi}{2} = \frac{-1}{2}\left(x - \frac{\pi}{4}\right) \Rightarrow x + 2y = \frac{5\pi}{4}$$

P3:

For the conic $\boldsymbol{ax^2 + by^2 = c}$, Find $\frac{d^2y}{dx^2}$.

Solution:

Given conic is $ax^2 + by^2 = c$

Differentiate with respect to x

$$2ax + 2by\frac{dy}{dx} = 0 \qquad \left(\therefore \frac{dy}{dx} = -\frac{ax}{by}\right)$$

Differentiate again with respect to x

$$a + b\left(\frac{dy}{dx}\right)^2 + by\frac{d^2y}{dx^2} = 0$$

$$\Longrightarrow \frac{d^2y}{dx^2} = \frac{a+b\left(\frac{dy}{dx}\right)^2}{by}$$

$$= -\frac{a+b\left(\frac{a^2}{b^2}\times\frac{x^2}{y^2}\right)}{by} = -\frac{a(by^2+ax^2)}{b^2y^3} = -\frac{ac}{b^2y^3}$$

P4:

If $x^3 + y^3 = 16$, then find the value of $\frac{d^2y}{dx^2}$ at $(2, 2)$

Solution:

$$x^3 + y^3 = 16$$

Differentiation with respect to x

$$x^2 + y^2 \frac{dy}{dx} = 0 \quad \left(\frac{dy}{dx} = -\left(\frac{x}{y}\right)^2\right)$$

Differentiation with respect to x

$$2x + 2y\left(\frac{dy}{dx}\right)^2 + y^2 \frac{d^2y}{dx^2} = 0$$

$$\Longrightarrow \frac{d^2y}{dx^2} = -\frac{\left(2x + 2y\left(-\left(\frac{x}{y}\right)^2\right)^2\right)}{y^2}$$

The value of $\frac{d^2y}{dx^2}$ at $(2,2) = -\frac{8}{4} = -2$

IP1:

Find the tangent and normal to the curve $\boldsymbol{x^2 + xy - y^2 = 1}$ at $\boldsymbol{(2,3)}$

Solution:

The given curve is $x^2 + xy - y^2 = 1$ and $(2,3)$ lies on it.

Differentiate with respect to x

$$2x + x\frac{dy}{dx} + y - 2y\frac{dy}{dx} = 0$$

$$\Longrightarrow (x - 2y)\frac{dy}{dx} + y + 2x = 0$$

$$\Longrightarrow \frac{dy}{dx} = \frac{-(2x+y)}{x-2y}$$

$$\therefore \text{slope} = \left.\frac{dy}{dx}\right|_{(2,3)} = \frac{-(4+3)}{2-6} = \frac{-7}{-4} = \frac{7}{4}$$

Tangent to the curve at $(2,3)$

$$y - 3 = \frac{7}{4}(x - 2) \Longrightarrow 4y - 12 = 7x - 14$$

$$\Longrightarrow 7x - 4y = 2$$

Slope of the normal to the curve at $(2,3) = -\frac{1}{\left(\frac{dy}{dx}\right)_{(2,3)}} = -\frac{4}{7}$

Normal to the curve at $(2,3)$

$$y - 3 = \frac{-4}{7}(x - 2) \Longrightarrow 4x + 7y - 29 = 0$$

IP2:

Find the tangent and normal to the curve $\boldsymbol{x^2\ cos^2y - \ siny = 0}$ at $(\mathbf{0},\ \boldsymbol{\pi})$

Solution:

The given curve is $x^2\ cos^2y - \ siny = 0$ and $(0,\ \pi)$ lies on it.

Differentiate both sides with respect to x

$$2x\ cos^2y + x^2.2\ cosy(-\ siny)\ \frac{dy}{dx} - cosy\frac{dy}{dx} = 0$$

$$\Rightarrow 2x\ cos^2y - x^2\ \ sin2y\ \frac{dy}{dx} - cosy\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x\ cos^2y}{(x^2\ sin2y + cosy)}$$

$$\therefore \text{slope} = \frac{dy}{dx}_{(0,\ \ \pi)} = \frac{2x\ .cos^2y}{x^2\ sin2y + cosy} = 0$$

Tangent to the curve at $(0,\ \pi)$ $y - \pi = 0 \Rightarrow y = \pi$

Slope of the normal to the curve at $(0,\ \pi) = -\frac{1}{\left(\frac{dy}{dx}\right)_{(0,\pi)}} = s$

where,$s \longrightarrow \infty$

the equation of the Normal to the curve at $(0,\ \pi)$

$$y - \pi = s(x - 0) \Rightarrow x - 0 = \frac{1}{s}(y - \pi) = 0, s \longrightarrow \infty$$

Thus, the normal to the curve at $(0,\ \pi)$ is $x = 0$ that is $y\_\ axis$

IP3:

If $\boldsymbol{ax^2 + 2hxy + by^2 = 1}$ **prove that** $\frac{d^2y}{dx^2} = \frac{h^2-ab}{(hx+by)^3}$

Solution:

$$ax^2 + 2hxy + by^2 = 1 \qquad \ldots (1)$$

Differentiate w. r. t. x, we get

$$2ax + 2h\left[x\frac{dy}{dx} + y.1\right] + 2by.\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(hy+ax)}{hx+by} \qquad \ldots (2)$$

$$\frac{d^2y}{dx^2} = -\left[\frac{(hx+by)\left(h\frac{dy}{dx}+a\right)-(hy+ax)\left(h+b\frac{dy}{dx}\right)}{(hx+by)^2}\right]$$

$$= \frac{-x(h^2-ab)\frac{dy}{dx}+(h^2-ab)y}{(hx+by)^2}$$

$$= \frac{h^2-ab}{(hx+by)^2}\left[y + x\left(\frac{hy+ax}{hx+by}\right)\right] \quad \text{using (2)}$$

$$= \frac{h^2-ab}{(hx+by)^3}(h\,xy + by^2 + h\,xy + ay^2)$$

$$= \frac{h^2-ab}{(hx+by)^3} \quad \text{using (1)}$$

$$\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx + by)^3}$$

IP4:

$\boldsymbol{xy + y^2 = 16}$, **Find the value of** $\frac{d^2y}{dx^2}$ **at** $(\mathbf{0}, -\mathbf{1})$

Solution:

$xy + y^2 = 16$

Differentiate with respect to x

$$x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0 \qquad \left(\therefore \frac{dy}{dx} = \frac{-y}{(x+2y)}\right)$$

$$\Rightarrow (x+2y)\frac{dy}{dx} + y = 0$$

Differentiate with respect to x

$$(x+2y)\frac{d^2y}{dx^2} + \left(1 + 2\frac{dy}{dx}\right)\frac{dy}{dx} + \frac{dy}{dx} = 0$$

$$\Rightarrow (x+2y)\frac{d^2y}{dx^2} + 2\left(1 + \frac{dy}{dx}\right)\frac{dy}{dx} = 0$$

$$\frac{d^2y}{dx^2} = -\frac{2\left(1+\frac{dy}{dx}\right)\frac{dy}{dx}}{(x+2y)} = \frac{2\left(1-\frac{y}{(x+2y)}\right)\left(\frac{y}{(x+2y)}\right)}{(x+2y)}$$

$$\frac{d^2y}{dx^2} = \frac{2y(x+y)}{(x+2y)^3}$$

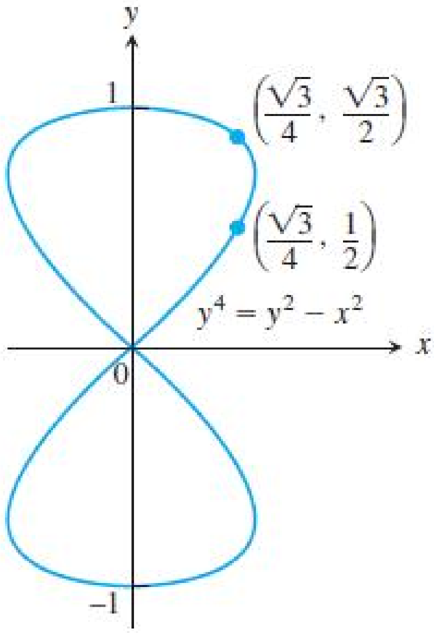
The value of $\frac{d^2y}{dx^2}$ at $(0, -1) = -\frac{1}{4}$

8.4. Tangents and Normal Lines

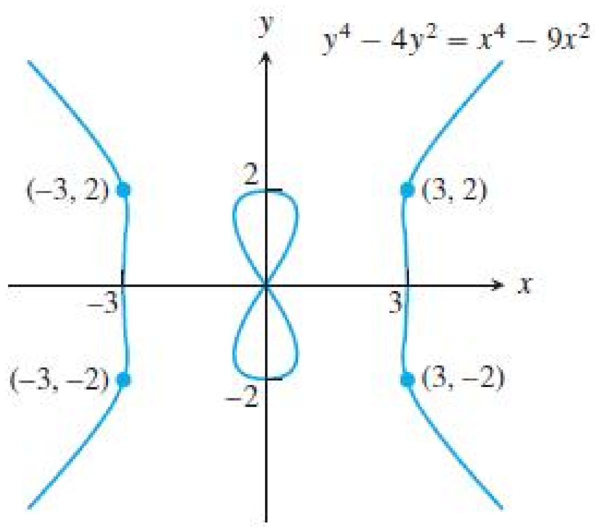
Exercise:

1. Find the lines that are (i) tangent and (ii) normal to the curve at the given point.
   a. $x^2 + xy - y^2 = 1$ $(2, 3)$
   b. $x^2y^2 = 9$ $(-1, 3)$
   c. $6x^2 + 3xy + 2y^2 + 17y - 6 = 0$ $(-1, 0)$
   d. $2xy + \pi \sin y = 2\pi$ $(1, \frac{\pi}{2})$
   e. $y = 2\sin(\pi - y)$ $(1, 0)$
2. Find second order differentiation to the following functions
   a. $x^2 + y^2 = 1$
   b. $x^2 + 2x = y^2$
   c. $y^2 - 2x = 1 - 2y$
   d. $y = \frac{1}{9} cot(3x - 1)$
   e. $y^3 + y = 2cosx$ at $(0,1)$
3. Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x-axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?

4. Find the slopes of the curve $y^4 = y^2 - x^2$ at the two points shown here.



5. Find the slopes of the devil's curve $y^4 - 4y^2 = x^4 - 9x^2$ at the four indicated points.



8.3

Implicit Differentiation

Learning objectives:

- To discuss implicit differentiation

  And

- To solve related problems.

Implicit Differentiation

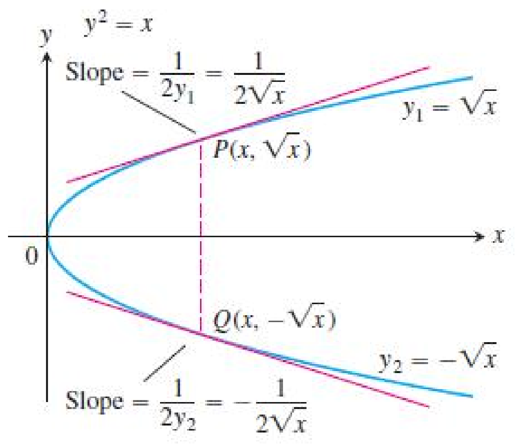
When we cannot put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate in the usual way, we may still be able to find $\frac{dy}{dx}$ by ***implicit differentiation***.
This consists of differentiation both sides of the equation with respect to x and then solve resulting equation for $\frac{dy}{dx}$.
The process by which we find $\frac{dy}{dx}$ is called ***implicit differentiation***.

Example 1: Find $\frac{dy}{dx}$ if $y^2 = x$

Solution

The equation $y^2 = x$ defines two differentiable functions of x

$$y_1 = \sqrt{x} \qquad y_2 = -\sqrt{x}$$



The derivative for each of these for $x > 0$ is

$$\frac{dy_1}{dx} = \frac{1}{2\sqrt{x}} \qquad \text{and} \qquad \frac{dy_2}{dx} = -\frac{1}{2\sqrt{x}}$$

We can find the derivatives for both by implicit differentiation. We simply differentiate both sides of the equation $y^2 = x$ with respect to x, treating $y = f(x)$ as a differentiable function of x:

$$y^2 = x \Rightarrow 2y\,\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$$

This one formula gives the derivatives we calculated for both of the explicit solutions $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$

$$\frac{dy_1}{dx} = \frac{1}{2y_1} = \frac{1}{2\sqrt{x}}\ , \quad \frac{dy_2}{dx} = \frac{1}{2y_2} = -\frac{1}{2\sqrt{x}}$$

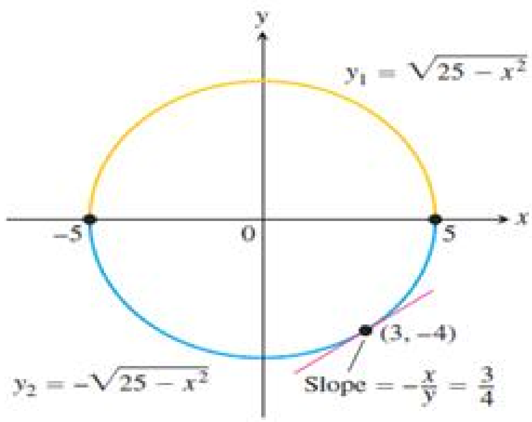
Example 2: Find the slope of circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution

The circle is not the graph of a single function of x. It is the combined graphs of two differentiable functions,

$y_1 = \sqrt{25 - x^2}$ and $y_2 = -\sqrt{25 - x^2}$. The point (3, −4) lies on the graph of y_2, so we can find the slope by calculating explicitly:

$$\frac{dy_2}{dx}\Big|_{x=3} = \frac{2x}{2\sqrt{25-x^2}}\Big|_{x=3} = \frac{6}{2\sqrt{25-9}} = \frac{3}{4}$$



We can also solve the problem more easily by differentiating the given equation of the circle implicitly with respect to x:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$\Rightarrow 2x + 2y\,\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

The slope at (3, −4) is $-\frac{x}{y}\big|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}$

The formula $\frac{dy}{dx} = -\frac{x}{y}$ applies *everywhere the circle has a slope*. The derivative involves both variables x and y, not just the independent variable x.

Implicit differentiation takes four steps.

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with $\frac{dy}{dx}$ on one side of the equation.
3. Factor out $\frac{dy}{dx}$.
4. Solve for $\frac{dy}{dx}$ by dividing.

Example 3: Find $\frac{dy}{dx}$ if $2y = x^2 + \sin y$.

Solution

$2y = x^2 + sin\, y$

Differentiate both sides with respect to x

$$\frac{d}{dx}(2y) = \frac{d}{dx}(x^2) + \frac{d}{dx}(siny)$$

$$\Rightarrow 2\frac{dy}{dx} = 2x + cosy\,\frac{dy}{dx} \Rightarrow 2\frac{dy}{dx} - cosy\,\frac{dy}{dx} = 2x$$

P1:

Find $\frac{dy}{dx}$ if $\boldsymbol{x^3 + y^3 = 3axy}$

Solution:

$$x^3 + y^3 = 3axy$$

We find $\frac{dy}{dx}$ through implicit differentiation

Differentiate both sides with respect to x,we get

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(3axy)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y\right]$$

$$\Rightarrow x^2 + y^2\frac{dy}{dx} = a\left[x\frac{dy}{dx} + y\right]$$

$$\Rightarrow (y^2 - ax)\frac{dy}{dx} = ay - x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

P2:

Find $\frac{dr}{d\theta}$ if $\boldsymbol{\theta^{\frac{1}{2}} + r^{\frac{1}{2}} = 1}$

Solution:

Given, $\theta^{\frac{1}{2}} + r^{\frac{1}{2}} = 1$

We find $\frac{dr}{d\theta}$ through implicit differentiation

Differentiate both sides with respect to θ

$$\frac{d}{d\theta}\left(\theta^{\frac{1}{2}}\right) + \frac{d}{d\theta}\left(r^{\frac{1}{2}}\right) = \frac{d}{d\theta}(1)$$

$$\Longrightarrow \frac{1}{2}\,\theta^{-\frac{1}{2}} + \frac{1}{2}\,r^{-\frac{1}{2}}\,\frac{dr}{d\theta} = 0$$

$$\Longrightarrow \theta^{-\frac{1}{2}} = -r^{-\frac{1}{2}}\,\frac{dr}{d\theta}$$

$$\Longrightarrow \frac{dr}{d\theta} = -\sqrt{\frac{r}{\theta}}$$

P3:

Find the slope of $\boldsymbol{x \sin 2y = y\ \cos 2x\,.\,at\ \left(\frac{\pi}{4},\ \frac{\pi}{2}\right)}$

Solution:

Given, $x \sin 2y = y\ \cos 2x.$

We find $\frac{dy}{dx}$ through implicit differentiation

Differentiate both sides with respect to x

$$\frac{d}{dx}(x \sin 2y) = \frac{d}{dx}(y\ \cos 2x)$$

$$\Longrightarrow x\ \cos 2y.2.\frac{dy}{dx} + \sin 2y = y.(-sin\ 2x).2 + cos\ 2x\frac{dy}{dx}$$

$$\Longrightarrow x\ \cos 2y.2.\frac{dy}{dx} + \sin 2y = (-sin2x)(2y) + cos\ 2x\ .\frac{dy}{dx}$$

$$\Longrightarrow (2x\ cos\ 2y - cos\ 2x)\frac{dy}{dx} = -2y \sin 2x - \sin 2y$$

$$\Longrightarrow \frac{dy}{dx} = -\frac{[\sin 2y + 2y \sin 2x]}{(2x \cos 2y - \cos 2x)}$$

$$slope = \left.\frac{dy}{dx}\right|_{\left(\frac{\pi}{4},\ \frac{\pi}{2}\right)} = -\frac{[sin\ (\pi) + \pi.(sin\ (\frac{\pi}{2}))]}{\left[\frac{\pi}{2} cos\ (\pi) - cos\ \ (\frac{\pi}{2})\right]}$$

$$= \frac{[0+\pi]}{\left[\frac{\pi}{2}(-1) - 0\right]} = -2$$

P4:

Find $\frac{dy}{dx}$ using implicit differentiation

$$\mathbf{6y^2 + 3xy + 2y^2 + 17y - 6 = 0}$$

Solution:

Given, $6y^2 + 3xy + 2y^2 + 17y - 6 = 0$

We find $\frac{dy}{dx}$ through implicit differentiation

Differentiate both sides with respect to x

$$\frac{d}{dx}(\ 6y^2) + \frac{d}{dx}(3xy) + \frac{d}{dx}(2y^2) + \frac{d}{dx}(17y) - \frac{d}{dx}(6) = 0$$

$$\Longrightarrow 12y\frac{dy}{dx} + 3x\frac{dy}{dx} + 3y + 4y\frac{dy}{dx} + 17\frac{dy}{dx} = 0$$

$$\Longrightarrow (12y + 4y + 3x + 17)\frac{dy}{dx} = -3y$$

$$\Longrightarrow \frac{dy}{dx} = \frac{-3y}{(16y + 3x + 17)}$$

IP1:

Find the derivative of $\boldsymbol{x + \sin y = xy}$

Solution:

Given, $x + \sin y = xy$

We find $\frac{dy}{dx}$ through implicit differentiation

Differentiate both sides with respect to x

$$\frac{d}{dx}(x) + \frac{d}{dx}(\sin y) = \frac{d}{dx}(xy)$$

$$\Rightarrow 1 + \cos y \frac{dy}{dx} = x\frac{dy}{dx} + y$$

$$\Rightarrow (\cos y - x)\frac{dy}{dx} = y - 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{y-1}{\cos y - x}$$

IP2:

Find $\frac{dr}{d\theta}$ if $\boldsymbol{\cos r + \sin \theta = r\,\theta}$

Solution:

Given, $\cos r + \sin \theta = r\,\theta$

We find $\frac{dr}{d\theta}$ through implicit differentiation

Differentiate both sides with respect to θ

$$\frac{d}{d\theta}(\cos r) + \frac{d}{d\theta}(\sin \theta) = \frac{d}{d\theta}(r\,\theta)$$

$$\Rightarrow -\sin r\,\frac{dr}{d\theta} + \cos\theta = r + \theta\frac{dr}{d\theta}$$

$$\Rightarrow \cos\theta - r = \frac{dr}{d\theta}(\theta + \sin r)$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{\cos\theta - r}{\theta + \sin r}$$

IP3:

Find the slope of $\boldsymbol{y = 2\,sin\,(\pi x - y)\;at\;(1, 0)}$

Solution:

Given, $y = 2\,sin\,(\pi x - y)$

We find $\frac{dy}{dx}$ through implicit differentiation

Differentiate both sides with respect to x

$$\frac{d}{dx}(y) = 2\frac{d}{dx}\left(sin\,(\pi x - y)\right)$$

$$\Rightarrow \frac{dy}{dx} = 2\,cos(\pi x - y).\left[\pi - \frac{dy}{dx}\right]$$

$$\Rightarrow \frac{dy}{dx} = 2\,cos(\pi x - y)\,\pi - 2\,cos(\pi x - y)\,\frac{dy}{dx}$$

$$\frac{dy}{dx}\,[1 + 2\,cos(\pi x - y)] = 2\,cos\,(\pi x - y)\pi$$

$$\frac{dy}{dx} = \frac{2\,cos\,(\pi x - y)\pi}{1 + 2\,cos(\pi x - y)}$$

$$\therefore\, slope = \left(\frac{dy}{dx}\right)_{(1,0)} = \frac{2\,cos\,(\pi).\pi}{1 + 2\,cos\pi} = \frac{-2\pi}{1 + 2(-1)} = \frac{-2\pi}{-1} = 2\pi$$

IP4:

If $\boldsymbol{siny = x\, sin(a + y)}$ then $\frac{dy}{dx} =$

Solution:

Given, $siny = x\, sin(a + y)$

We find $\frac{dy}{dx}$ through implicit differentiation

Differentiate both sides with respect to x

$$\frac{d}{dx}(siny) = \frac{d}{dx}(x\, sin(a + y))$$

$$\Rightarrow cosy\frac{dy}{dx} = x\ \frac{d}{dx}\sin(a + y) + sin(a + y)$$

$$\Rightarrow cosy\frac{dy}{dx} = x\cos(a + y)\frac{dy}{dx} + sin(a + y)$$

$$\Rightarrow cosy\frac{dy}{dx} = \left(\frac{siny}{\sin(a+y)}\right)\ \cos(a + y)\frac{dy}{dx} + sin(a + y)$$

$$since\ x = \left(\frac{siny}{\sin(a+y)}\right)$$

$$\Rightarrow (cosy\sin(a + y) - siny\cos(a + y))\frac{dy}{dx} = sin^2(a + y)$$

$$\Rightarrow \sin(a + y - y)\frac{dy}{dx} = sin^2(a + y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{sin^2(a + y)}{sina}$$

8.3. Implicit Differentiation

Exercise:

I. Use implicit differentiation to find $\frac{dy}{dx}$.

a. $x^2y + xy^2 = 6$

b. $x^3 + y^3 = 18xy$

c. $2xy + y^2 = x + y$

d. $x^3 - xy + y^3 = 1$

e. $x^2(x - y)^2 = x^2 - y^2$

f. $(3xy + 7)^2 = 6y$

g. $y^2 = \frac{x-1}{x+1}$

h. $x^2 = \frac{x-y}{x+y}$

II. Use implicit differentiation to find $\frac{dy}{dx}$.

a. $x = tany$

b. $xy = \cot(xy)$

c. $x + \tan(xy) = 0$

d. $x + siny = xy$

e. $ysin\left(\frac{1}{y}\right) = 1 - xy$

f. $y^2 \cos\left(\frac{1}{y}\right) = 2x + 2y$

III. Use implicit differentiation to find $\frac{dr}{d\theta}$.

a. $\sin(r\theta) = \frac{1}{2}$

b. $cosr + cot\theta = r\theta$

IV. Find the slope of the following functions:

a. $6y^2 + 3xy + 2y^2 + 17y - 6 = 0, \quad (-1, 0)$

b. $x^2 - \sqrt{3xy} + 2y^2 = 5, \quad (\sqrt{3}, 2)$

c. $2xy + \pi\ siny = 2\pi, \quad \left(1, \frac{\pi}{2}\right)$

d. $x^2 cos^2 y - siny = 0, \quad (0, \pi)$

8.5

Rational Powers of Differential Functions

Learning objectives:

- To extend the power rule for differentiation to rational exponents through implicit differentiation

  And
- To practice related problems.

Rational Powers of Differential Functions

This module uses the technique of implicit differentiation to extend the Power Rule for differentiation to include all rational exponents.

The power rule $\frac{d}{dx}x^n = nx^{n-1}$

was proved to hold when n is an integer. We can now show that it holds when n is any rational number.

Theorem: Power rule for differentiation for rational numbers

If n is a rational number, then x^n is differentiable at every interior point x of the domain of x^{n-1}, and

$$\frac{d}{dx}x^n = nx^{n-1}$$

Proof:

Let $n = \frac{p}{q}$ where p and q are integers with $q \neq 0$ so that $y = x^n = x^{\frac{p}{q}}$. Then $y^q = x^p$

This equation is an algebraic combination of powers of x and y, and y is a differentiable function of x. Since p and q are integers, we can differentiate both sides of the equation implicitly with respect to x and obtain $q\,y^{q-1}\frac{dy}{dx} = p\,x^{p-1}$

We solve for $\frac{dy}{dx}$ and

$$\frac{dy}{dx} = \frac{px^{p-1}}{qy^{q-1}}$$

$$= \frac{p}{q} \cdot \frac{x^{p-1}}{(x^{p/q})^{q-1}}$$

$$= \frac{p}{q} \cdot \frac{x^{p-1}}{x^{p-p/q}} \Longrightarrow = \frac{p}{q} \cdot x^{p-1-p+p/q}$$

$$= \frac{p}{q} \cdot x^{\left(\frac{p}{q}\right)-1}$$

$$= nx^{n-1}$$

This proves the rule.

Example 1:

a) $\frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}\, x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$; function is defined for $x \geq 0$, derivative is defined only for $x > 0$.

b) $\frac{d}{dx}\left(x^{\frac{1}{5}}\right) = \frac{1}{5}\, x^{-\frac{4}{5}}$; function is defined for all x, derivative is not defined at $x = 0$

In terms of the Chain rule, the Power Rule states that if n is a rational number, u is differentiable at x, and u^{n-1}

is defined, then u^n is differentiable at x, and

$$\frac{d}{dx} u^n = nu^{n-1} \frac{du}{dx}$$

Example 2:

a) $\frac{d}{dx}(1-x^2)^{\frac{1}{4}} = \frac{1}{4}\,(1-x^2)^{-\frac{3}{4}}\,(-2x) = \frac{-x}{2(1-x^2)^{\frac{3}{4}}}$;

Notice that the function is defined on [−1, 1], but the derivative is defined only on (−1, 1).

b) $\frac{d}{dx}(\cos x)^{-\frac{1}{5}} = -\frac{1}{5}\,(\cos x)^{-\frac{6}{5}}\,\frac{d}{dx}\,\cos x$

$$= -\frac{1}{5}\,(\cos x)^{-\frac{6}{5}}\,(-\sin x)$$

$$= \frac{1}{5}\,\sin x\,(\cos x)^{-\frac{6}{5}}$$

P1:

Find $\frac{dy}{dx}$ if $\boldsymbol{y = x^{\frac{-3}{5}}}$

Solution:

$$y = x^{\frac{-3}{5}}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{\frac{-3}{5}}\right) = \left(-\frac{3}{5}\right)x^{-\frac{1}{5}-1} = -\frac{3}{5}x^{-\frac{6}{5}}$$

Function is defined for all $x \in \boldsymbol{R}$, $x \neq 0$, derivative is not defined at $x = 0$

P2:

Find $\frac{dy}{dx}$ if $\boldsymbol{y = (1 - 6x)^{\frac{2}{3}}}$

Solution:

$$y = (1 - 6x)^{\frac{2}{3}}$$

$$\frac{dy}{dx} = \frac{d}{dx}(1 - 6x)^{\frac{2}{3}} = \frac{2}{3}(1 - 6x)^{\frac{2}{3}-1}.\frac{d}{dx}(1 - 6x)$$

$$= \frac{2}{3}(1 - 6x)^{-\frac{1}{3}}.(-6) = \frac{-4}{(1-6x)^{\frac{1}{3}}}$$

Function is defined for all x, derivative is defined for all

$x\ except\ x \neq \frac{1}{6}.$

P3:

Find the derivative of $\boldsymbol{y = (\sin(\theta + 5))^{\frac{5}{4}}}$

Solution:

$$y = (\sin(\theta + 5))^{\frac{5}{4}}$$

$y = u^{\frac{5}{4}}$, where $u(v) = \sin v$ and $v(\theta) = \theta + 5$

Now, $\frac{dy}{du} = \frac{5}{4}u^{\frac{1}{4}}$, $\frac{du}{dv} = \cos v$ and $\frac{dv}{d\theta} = 1$

By chain rule $\frac{dy}{d\theta} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{d\theta}$

$$\frac{dy}{d\theta} = \frac{5}{4}u^{\frac{1}{4}}.\cos v\,.\,1$$

$$= \frac{5}{4}(\sin(\theta + 5))^{\frac{1}{4}}.\cos(\theta + 5)\,.\,1$$

$$= \frac{5}{4}(\sin(\theta + 5))^{\frac{1}{4}}.\cos(\theta + 5)$$

P4:

Find the derivative of $\boldsymbol{y = \sqrt[3]{1 + cos2\theta}}$

Solution:

$$y = \sqrt[3]{1 + cos2\theta}$$

Now, $y = u^{\frac{1}{3}}$ where $u = 1 + cos2\theta$

By chain rule

$$\frac{dy}{d\theta} = \frac{dy}{du} \cdot \frac{du}{d\theta}$$

$$= \frac{d}{du}\left(u^{\frac{1}{3}}\right) \times \frac{d}{d\theta}(1 + cos2\theta) = \frac{1}{3}u^{-\frac{2}{3}}(-sin2\theta)(2)$$

$$= -\frac{2}{3}(1 + cos2\theta)^{-\frac{2}{3}}sin2\theta$$

IP1:

Find $\frac{dy}{dx}$ if $\boldsymbol{y = \sqrt[4]{5x}}$

Solution:

$$y = \sqrt[4]{5x}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left((5x)^{\frac{1}{4}}\right) = \frac{1}{4}(5x)^{\frac{1}{4}-1} = \frac{1}{4}(5x)^{-\frac{3}{4}}$$

Function is defined for all $x \geq 0$, derivative is not defined at $x = 0$

IP2:

Find $\frac{dy}{dx}$ if $y = x(x^2 + 1)^{\frac{1}{2}}$

Solution:

$$y = x(x^2 + 1)^{\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{dx} = x\frac{d}{dx}(x^2 + 1)^{\frac{1}{2}} + (x^2 + 1)^{\frac{1}{2}}\frac{d}{dx}(x)$$

$$= x\frac{1}{2}(x^2 + 1)^{\frac{1}{2}-1}\frac{d}{dx}(x^2 + 1) + (x^2 + 1)^{\frac{1}{2}}.1$$

$$= 2x^2\ \frac{1}{2\sqrt{x^2+1}} + \sqrt{x^2 + 1}$$

$$= \frac{x^2+x^2+1}{\sqrt{(x^2+1)}} = \frac{(2x^2+1)}{\sqrt{(x^2+1)}}$$

IP3:

Find the derivative of $\boldsymbol{y = \cos\left((1-6t)^{\frac{2}{3}}\right)}$

Solution:

$$y = \cos\left((1-6t)^{\frac{2}{3}}\right)$$

$y = \cos u$, where $u = v^{\frac{2}{3}}$ and $v(t) = 1 - 6t$

Now $\frac{dy}{du} = -\sin u$, $\frac{du}{dv} = \frac{2}{3} v^{\frac{-1}{3}}$ and $\frac{dv}{dt} = -6$

By chain rule $\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dt}$

$$\frac{dy}{dt} = -\sin u . \frac{2}{3} v^{-\frac{1}{3}} . (-6)$$

$$= 4\sin\left((1-6t)^{\frac{2}{3}}\right) . (1-6t)^{-\frac{1}{3}}$$

IP4:

Find the derivative of the function given by $\boldsymbol{y = \sin\left[(2t+5)^{-\frac{2}{3}}\right]}$

Solution:

$$y = \sin\left[(2t+5)^{-\frac{2}{3}}\right]$$

Now , $y = \sin u,\ u = v^{-\frac{2}{3}}$ and $v = (2t+5)$

$$\frac{dy}{du} = \cos u,\ \frac{du}{dv} = \frac{-2}{3}v^{\frac{1}{3}},\ \frac{dv}{dt} = 2$$

By chain rule

$$\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dt}$$

$$\frac{dy}{dt} = \cos u\left[-\frac{2}{3}v^{1/3}.2\right]$$

$$= -\frac{4}{3}\cos\left((2t+5)^{-\frac{2}{3}}\right).(2t+5)^{1/3}$$

8.5. Rational Powers of Differential Functions

Exercise:

1. State and prove power rule for differentiation for rational numbers.
2. Find $\frac{dy}{dx}$.
   a. $y = x^{9/4}$
   b. $x = \sqrt[3]{2x}$
   c. $y = 7\sqrt{x+6}$
   d. $y = (2x+5)^{-\frac{1}{2}}$
   e. $y = x(x^2+1)^{\frac{1}{2}}$
3. Find the first derivatives of the functions.
   a. $s = \sqrt[7]{t^2}$
   b. $f(x) = \sqrt{1-\sqrt{x}}$
   c. $r = \frac{3}{2}\theta^{\frac{2}{3}} + \frac{4}{3}\theta^{\frac{3}{4}}$

8.6

Related Rates of Change

Learning objectives:

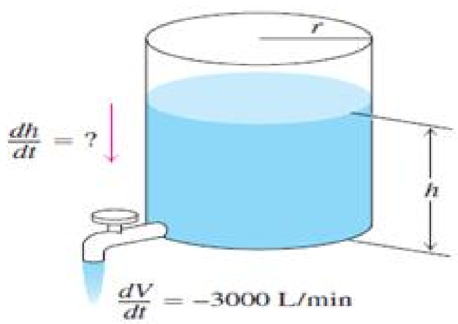
- To practice some related rates problems.

The problem of finding a rate, which can't be measured easily, from some other measurable rates, is called a *related rates problem*.

When we pump the fluid out stored in a vertical cylindrical storage tank, the fluid level inside the tank drops down. In general it is difficult to measure the rate of change in the level of fluid. However it can be determined from the rate of pumping which can be easily measured. We write an equation that relates the variables involved and differentiate to get an equation that relates the rate we seek to the rate we know.

Example 1:
How rapidly will the fluid level inside a vertical cylindrical tank drop if we pump the fluid out at the rate of 3000 L/min?

Solution
Let r be the radius of the cylinder, h the height of the fluid, and V the volume.



As time passes, the radius remains constant, but V and h change. We think of V and h as differentiable functions of time and use t to represent time. It is given that

$$\frac{dV}{dt} = -3000$$

Since we pump out at the rate of 3000 L/min. The rate is negative because the volume is decreasing.

We are asked to find $\frac{dh}{dt}$: How fast will the fluid level drop?

Rates of change are represented by derivatives.

To find $\frac{dh}{dt}$, we first write an equation that relates h to V.

If V measured in litres and r, h in metres then $V = \pi r^2 h$ cubic metres. Since one cubic metre contains $1000L$, we write
$V = 1000\pi r^2 h$

We differentiate both sides of the equation with respect to time: $\frac{dv}{dt} = 1000\pi r^2 \frac{dh}{dt}$, r is a constant.

We substitute the known value $\frac{dv}{dt} = -3000$ and solve for $\frac{dh}{dt}$:

$$\frac{dh}{dt} = \frac{-3000}{1000\pi r^2} = -\frac{3}{\pi r^2}$$

The fluid level will drop at the rate of $\frac{3}{\pi r^2}$ m/min. The rate at which the fluid level drops depends on the tank's radius. If r is small, $\frac{dh}{dt}$ will be large; if r is large, $\frac{dh}{dt}$ will be small.

If $r = 1\ m$: $\frac{dh}{dt} = -\frac{3}{\pi} \approx -0.95\ m/min = -95\ cm/min$

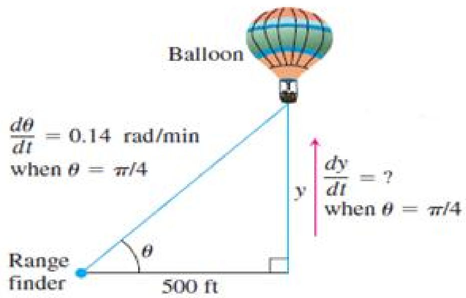
If $r = 10\ m$: $\frac{dh}{dt} = -\frac{3}{100\pi} \approx -0.0095 m/min = -0.95 cm/min$

Example 2:

A hot-air balloon rising straight up from a level field is tracked by a range finder 500 ft from the lift-off point. At the moment the range finder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

Solution

We draw a picture and name the variables and constants.



The variables are: θ is the angle the range finder makes with the ground (radians), y is the height of the balloon

We assume θ and y to be differentiable functions of time t.

Write down the numerical information.

$$\frac{d\theta}{dt} = 0.14\ rad/min \qquad when\ \theta = \frac{\pi}{4}$$

Write down what we are asked to find

We want $\frac{dy}{dt}$ when $\theta = \frac{\pi}{4}$.

Write an equation that relates the variables y and θ.

$$y = 500 \tan\theta$$

Differentiate with respect to t using the Chain Rule.

The result tells how $\frac{dy}{dt}$ (which we want) is related to $\frac{d\theta}{dt}$ (which we know).

$$\frac{dy}{dt} = 500\ sec^2\theta\,\frac{d\theta}{dt}$$

Evaluate with $\theta = \frac{\pi}{4}$ *and* $\frac{d\theta}{dt} = 0.14$ to find $\frac{dy}{dt}$.

$$\frac{dy}{dt} = 500\left(\sqrt{2}\right)^2(0.14) = 140$$

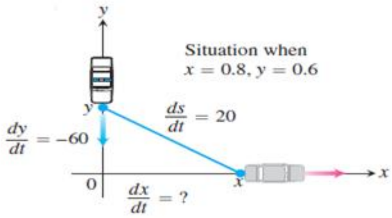
At the moment in question, the balloon is rising at the rate of 140 ft/min.

Example 3:

A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is $0.6\ km$ north of the intersection and the car is 0.8 km to the east, the police determine with radar that the distance between them and the car is increasing at $20\ km/hour$. If the cruiser is moving at $60\ km/hour$ at the instant of measurement, what is the speed of the car?

Solution

We picture the car and cruiser in the coordinate plane, using the positive x-axis as the eastbound highway and the positive y-axis as the northbound highway.



We let t represent time and set

x= position of car at time t

y= position of cruiser at time t

s = distance between car and cruiser at time t

We assume $x, y,$ and s are differentiable functions of t.

At the instant in question,

$$x = 0.8\ km, y = 0.6\ km, \frac{ds}{dt} = 20km/h, \quad \frac{dy}{dt} = -60\ km/h$$

$\frac{dy}{dt}$ is negative because y is decreasing.

We need to find $\frac{dx}{dt}$.

The variables are related by $s^2 = x^2 + y^2$

We differentiate with respect to t.

$$s\frac{ds}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = \frac{1}{\sqrt{x^2+y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

We evaluate with

$x = 0.8\ km, y = 0.6\ km, \frac{ds}{dt} = 20km/h, \frac{dy}{dt} = -60km/$ and solve for $\frac{dx}{dt}$.

$$20 = \frac{1}{\sqrt{0.8^2+0.6^2}}\left(0.8\ \frac{dx}{dt} + (0.6)(-60)\right) = 0.8\frac{dx}{dt} - 36$$

$$\frac{dx}{dt} = \frac{20+36}{0.8} = 70$$

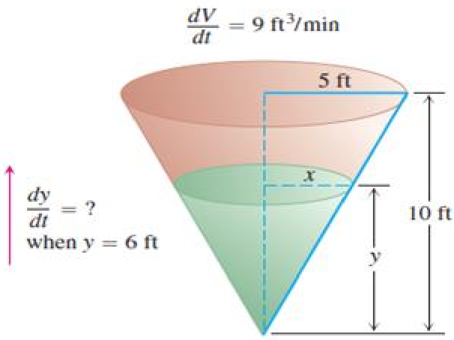
At the moment in question, the car's speed is 70 km/h.

Example 4:

Water runs into a conical tank at the rate of $9\ ft^3/min$. The tank stands point down and has a height of $10\ ft$ and a base radius of $5\ ft$. How fast is the water level rising when water is $6\ ft$ deep?

Solution

We draw a picture of a partially filled conical tank.



The variables in the problem are

V = volume (ft^3) of water in the tank at time t (min),

x= radius (ft) of the surface of water at time t,

y = depth (ft) of water in the tank at time t.

We wish to find $\frac{dy}{dt}$.

The variables are related by $V = \frac{1}{3}\pi x^2 y$.

The equation involves x also. Because no information is given about x and $\frac{dx}{dt}$ at the time in question, we need to eliminate x. From similar triangles,

$$\frac{x}{y} = \frac{5}{10} \qquad x = \frac{y}{2}$$

Therefore, $V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y = \frac{\pi}{12}y^3$

We differentiate with respect to t.

$$\frac{dV}{dt} = \frac{\pi}{12}.3y^2\frac{dy}{dt} = \frac{\pi}{4}y^2\frac{dy}{dt}$$

We then solve for $\frac{dy}{dt}$: $\frac{dy}{dt} = \frac{4}{\pi y^2}\frac{dV}{dt}$

We evaluate with $y = 6\ and\ \frac{dV}{dt} = 9$..

$$\frac{dy}{dt} = \frac{4}{\pi(6)^2}.9 = \frac{1}{\pi} \approx 0.32$$

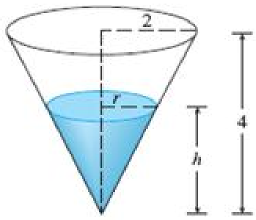
At the moment in question, the water level is rising at about $0.32\ ft/min$.

P1:

A water tank has the shape of an inverted circular cone with base radius $2m$ and height $4m$. If water is being pumped into the tank at a rate of $2m^3/min$, find the rate at which the water level is rising when the water is $3m$ deep.

Solution:

Let V, r and h be the volume of the water, the radius of the surface, and the height of the water at time t, where t is measured in minutes.



Given, $dV/dt = 2m^3/min$

We are asked to find $\frac{dh}{dt}$ when h is $3m$.

The quantities V and h are related by the equation

$$V = \frac{1}{3}\pi r^2 h$$

To express V as a function of h alone.

In order to eliminate r, we use the similar triangles to write

$$\frac{r}{h} = \frac{2}{4} \qquad r = \frac{h}{2}$$

The expression for V becomes

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

Differentiate both sides with respect to t:

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$

Substituting $h = 3\ m$ and $\frac{dV}{dt} = 2m^3/min$,

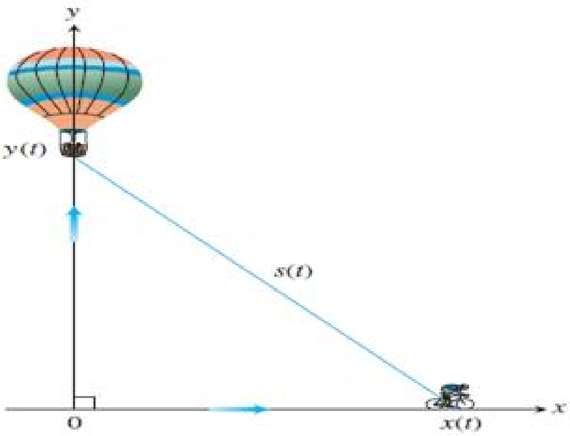
$$\frac{dh}{dt} = \frac{4}{\pi(3)^2}.2 = \frac{8}{9\pi}m/min$$

The water level is rising at a rate of $\frac{8}{9\pi}m/min \approx 0.28m/min$.

P2:

A balloon is rising vertically above a level, straight road at a constant rate of 1 ft/sec. Just when the balloon is 65 ft above the ground, a bicycle moving at a constant rate of 17 ft/sec passes under it. How fast is the distance $s(t)$ between the bicycle and balloon increasing 3 sec later?

Solution:



Here $y(t) =$ balloons height from O.

$x(t) =$ bicycle distance from O.

$S(t) =$ distance between balloon and bicycle

$t = 3$ sec

$\frac{ds}{dt} = ?$

From the given data, $s^2(t) = y^2(t) + x^2(t)$

$$\Rightarrow 2s(t)\frac{ds}{dt} = 2y(t)\frac{dy}{dt} + 2x(t)\frac{dx}{dt}$$

$$\Rightarrow s(t)\frac{ds}{dt} = y(t)\frac{dy}{dt} + x(t)\frac{dx}{dt}$$

$$\Rightarrow \frac{ds}{dt} = \frac{1}{s(t)}\left[y(t)\frac{dy}{dt} + x(t)\frac{dx}{dt}\right]$$

$$y(t) = 65 + 3 = 68\ ft, x(t) = 51\ ft$$

$$s(t) = \sqrt{(68)^2 + (51)^2} = 85$$

$$\frac{ds}{dt} = \frac{1}{85}[68(1) + 51(17)] = 11 ft/sec$$

P3:

Car A is traveling west at $50\ mi/h$ and car B is traveling north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 and car B is $0.4\ mi$ from the intersection?

Solution:

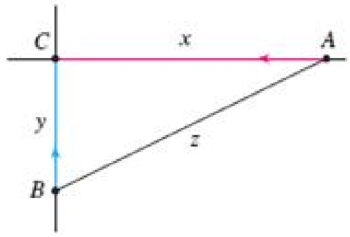


FIGURE 4

Here C is the intersection of the roads.

At a given time t, let x be the distance from car A to C, y be the distance from car B to C, and z be the distance between the cars, where $x, y,$ and z are measured in miles.

Given, $\frac{dx}{dt} = -50mi/h$ and $\frac{dy}{dt} = -60mi/h.$

To find $\frac{dz}{dt}$

The equation that relates x, y and z is given by the Pythagorean Theorem: $z^2 = x^2 + y^2$

Differentiating both sides with respect to t, we have

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

When $x = 0.3\ mi$ and $y = 0.4\ mi$, $z = 0.5\ mi$, so

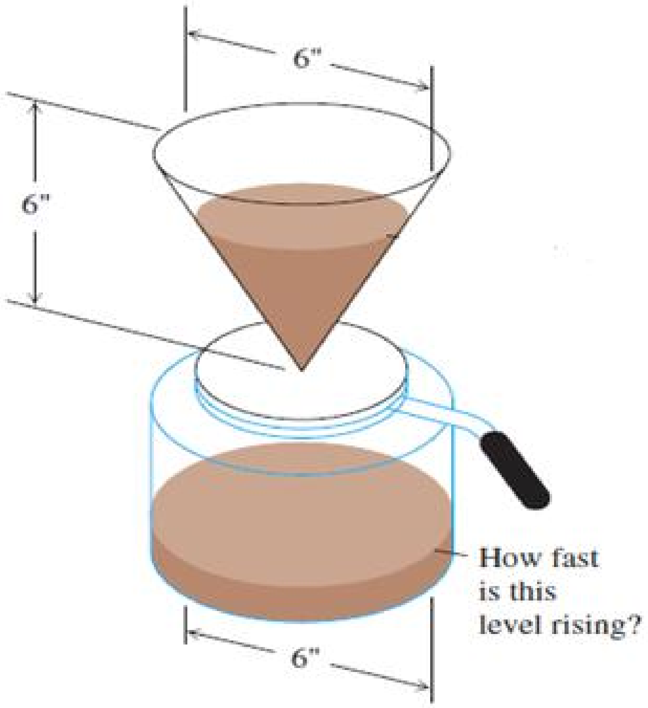
$$\frac{dz}{dt} = \frac{1}{0.5}[0.3(-50) + 0.4(-60)]$$

$$= -78\ mi/h$$

The cars are approaching each other at a rate of 78 mi/h.

P4:

Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of $10\ inch^3/min$. How fast is the level in the pot rising when the coffee in the cone is 5 inch. deep?



Solution:

Radius of cylindrical coffee pot $= 3\ inch$

Let h be the height of the coffee in the pot

Volume of the coffee $V = \pi r^2 h = 9\pi h$

$$\frac{dV}{dt} = 9\pi \frac{dh}{dt}$$

The rate the coffee is rising is

$$\frac{dh}{dt} = \frac{1}{9\pi}\frac{dV}{dt}$$

$$\frac{dh}{dt} = \frac{1}{9\pi}[10]$$

$$= \frac{10}{9\pi}\ inch/min$$

IP1:

A ladder 10ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Solution :

Let x feet be the distance from the bottom of the ladder to the wall and y feet be the distance from the top of the ladder to the ground. Note that x and y are both functions of t (time, measured in seconds).

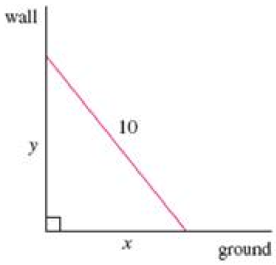


FIGURE 1

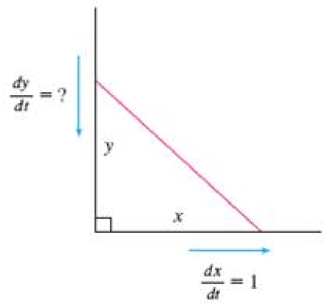


FIGURE 2

Given $\frac{dx}{dt} = 1\,ft/s$ and

Find $\frac{dy}{dt}$ when $x = 6\,ft$

The relationship between x and y is given by the Pythagorean Theorem: $x^2 + y^2 = 100$

Differentiating each side with respect to t using the chain Rule, we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Longrightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

When $x = 6$, the Pythagorean theorem gives $y = 8$ and so, substituting these values and $\frac{dx}{dt} = 1$, we get

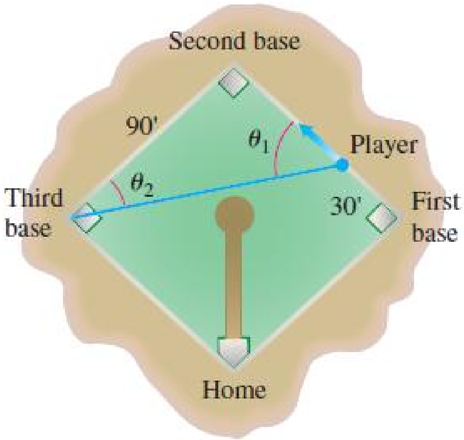
$$\frac{dy}{dt} = -\frac{6}{8}(1) = -\frac{3}{4}\ ft/s$$

The fact that $\frac{dy}{dt}$ is negative means that the distance from the top of the ladder to the ground is decreasing at a rate of $\frac{3}{4}ft/s$. In other words, the top of the ladder is sliding down the wall at a rate of $\frac{3}{4}ft/s$.

IP2:

A baseball diamond is a square 90 ft on a side. A player runs from first base to second at a rate of 16 ft/sec. At what rate is the player's distance from third base changing when the player is 30 ft from first base?

Solution:



$s(t) =$ Distance of the player from third base

$x(t) =$ Distance of the player from second base

$\frac{dx}{dt} = -16\ ft/sec$

From the image, $s^2 = x^2 + 8100$

$$\Rightarrow 2s\frac{ds}{dt} = 2x\frac{dx}{dt}$$

$$\Rightarrow s\frac{ds}{dt} = x\frac{dx}{dt} \qquad \ldots(1)$$

When the player is 30 ft from 1st base then $x = 60$

$$S = \sqrt{(90)^2 + (60)^2} = 30\sqrt{13}$$

From (1)

$$\frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt} = \frac{60}{30\sqrt{3}}(-16)$$

$$= -\frac{32}{\sqrt{3}} \approx -8.875\ ft/sec$$

IP3:

A man walks along a straight path at a speed of $4ft/s$. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the search light rotating when the man is 15 ft from the point on the path closest to the searchlight?

Solution:

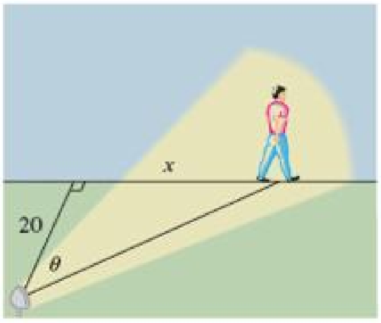


FIGURE 5

let x be the distance from the man to the point on the path closest to the searchlight. θ be the angle between the beam of the searchlight and the perpendicular to the path.

Given , $\frac{dx}{dt} = 4ft/s$

Find $\frac{d\theta}{dt}$ when $x = 15$.

The equation that relates x and θ is

$$\frac{x}{20} = tan\theta \Longrightarrow x = 20\, tan\theta$$

Differentiating both sides with respect to t, we get

$$\frac{dx}{dt} = 20sec^2\theta \frac{d\theta}{dt}$$

So $\quad \frac{d\theta}{dt} = \frac{1}{20} cos^2\theta \frac{dx}{dt} = \frac{1}{20} cos^2\theta(4) = \frac{1}{5} cos^2\theta$

When $x = 15$, the length of the beam is 25, so $cos\theta = \frac{4}{5}$ and

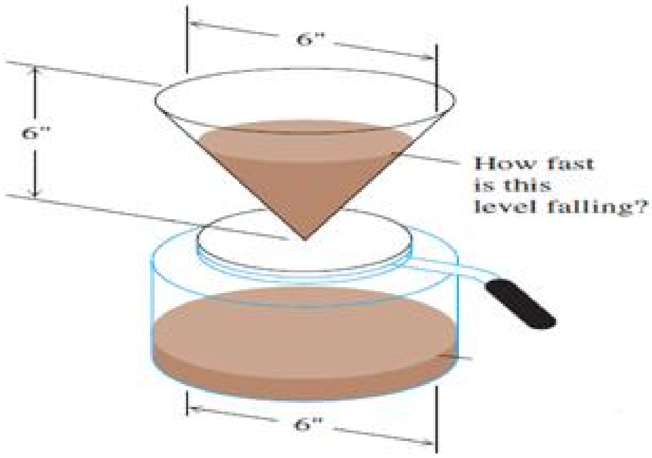
$$\frac{d\theta}{dt} = \frac{1}{5}\left(\frac{4}{5}\right)^2 = \frac{16}{125} = 0.128$$

The searchlight is rotating at a rate of $0.128\ rad/s$.

IP4:

Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of $\mathbf{10}\ \boldsymbol{inch^3/min}$. How fast is the level in the cone falling, when the coffee in the cone is 5 inch. deep?

Solution:



Let h be the height of the coffee in the cone.

From the figure, the radius of the cone $r = \frac{h}{2}$

Volume of the cone $V = \frac{1}{3}\pi r^2 h = \frac{1}{12}\pi h^3$

$$\Rightarrow \frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

When $h = 5\ inch$, then

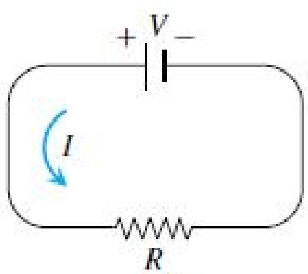
$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt} \Rightarrow \frac{dh}{dt} = \frac{4}{\pi(25)}(-10) = -\frac{8}{5\pi}\ inch/min$$

The rate the coffee level *falling* in the cone is $\frac{8}{5\pi}\ inch/min$

8.6. Related Rates of Change

Exercise:

1. Suppose that the radius r and area $A = \pi r^2$ of a circle are differentiable functions of t. Write an equation that relates $\frac{dA}{dt}$ to $\frac{dr}{dt}$.
2. The radius r and height h of a right circular cylinder are related to the cylinder's volume V by the formula $V = \pi r^2 h$.
   a. How is $\frac{dV}{dt}$ related to $\frac{dh}{dt}$ if r is constant?
   b. How is $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ if h is constant?
   c. How is $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ and $\frac{dh}{dt}$ if neither r nor h is constant?
3. The voltage V (volts), current I (amperes) and resistance R (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 volt/sec while I is decreasing at the rate of $\frac{1}{3}$ amp/sec. Let t denote time in seconds.



   a. What is the value of $\frac{dV}{dt}$?
   b. What is the value of $\frac{dI}{dt}$?
   c. What equation relates $\frac{dR}{dt}$ to $\frac{dV}{dt}$ and $\frac{dI}{dt}$?
   d. Find the rate at which R is changing when $V = 12$ volts and $I = 2$ amperes. Is R increasing, or decreasing?

4. Let x and $y$be differentiable functions of t and let $s = \sqrt{x^2 + y^2}$ to be the distance between the points $(x, 0)$ and $(0, y)$ in the xy-plane.

   a. How is $\frac{ds}{dt}$ related to $\frac{dx}{dt}$ if y is constant?

   b. How is $\frac{ds}{dt}$ related to $\frac{dx}{dt}$ and $\frac{dy}{dt}$ if neither x nor y is constant?

   c. How is $\frac{dx}{dt}$ related to $\frac{dy}{dt}$ if s is constant?

5. The area A of a triangle with sides of length a and b enclosing an angle of measure θ is

$$A = \frac{1}{2}\,ab\sin\theta$$

   a. How is $\frac{dA}{dt}$ related to $\frac{d\theta}{dt}$ if a and b are constant?

   b. How is $\frac{dA}{dt}$ related to $\frac{d\theta}{dt}$ and $\frac{da}{dt}$ if only b is constant?

   c. How is $\frac{dA}{dt}$ related to $\frac{d\theta}{dt}, \frac{da}{dt}$ and $\frac{db}{dt}$ if none of a, b, and θ are constant?

6. The length l of a rectangle is decreasing at the rate of 2 cm/sec while the width w is increasing at the rate of 2 cm/sec. When l = 12 cm and w = 5 cm, find the rates of change of

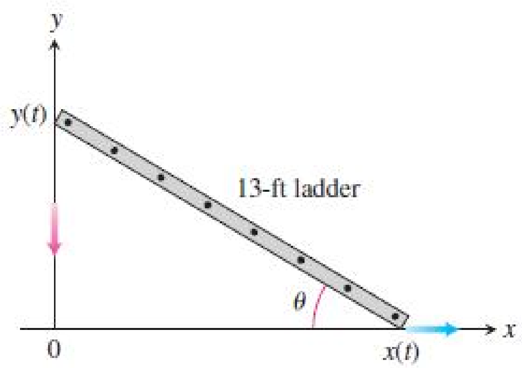
   a. the area,

   b. the perimeter, and

   c. the lengths of the diagonals of the rectangle.

   Which of these quantities are decreasing, and which are increasing?

7. A $13 - ft$ ladder is leaning against a house when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.

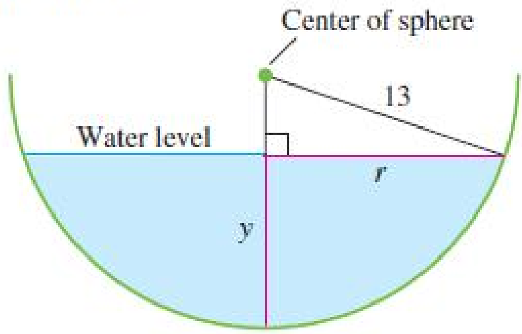


a. How fast is the top of the ladder sliding down the wall then?
b. At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?
c. At what rate is the angle θ between the ladder and the ground changing then?

8. A girl flies a kite at a height of 100 m, the wind carrying the kite horizontally away from her at a rate of 10 m/sec. How fast must she let out the string when the kite is 150 m away from her?
9. Sand falls from a conveyor belt at the rate of 10 m^3/min onto the top of a conical pile. The height of the pile is

   always three-eighths of the base diameter. How fast are the
   a. height and
   b. radius changing when the pile is 4 m high?
   Answer in cm/min.

10. Water is flowing at the rate of $6\ m^3/min$ from a reservoir shaped like a hemispherical bowl of radius 13 m, shown here in profile.



Answer the following questions, given that the volume of water in a hemispherical bowl of radius R is $V = \frac{\pi}{3}y^2(3R - y)$ when the water is y units deep.
   a. At what rate is the water level changing when the water is 8 m deep?
   b. What is the radius r of the water's surface when the water is y m deep?
   c. At what rate is the radius r changing when the water is 8 m deep?